lar or $TE_{111}°$ circular waveguide networks through the application of the mixed mode filter. A novel feature of this concept is its potential capability of realizing planar cross-coupled filter designs.

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Letters

The Admittance Matrix of Coupled Transmission Lines

A. I. GRAYZEL

***Abstract*—An alternate derivation of the admittance matrix of n coupled lines is presented. The method uses the superposition of two modes of excitation analagous to the even and odd mode excitation used for the analysis of two coupled lines. Only mutual capacitance to adjacent lines is considered.**

In this letter an alternate derivation of the admittance matrix derived by Riblet [1] is presented. The configuration is shown in Fig. 1 and the matrix is defined by (1) which are identical to [1, fig. 1 and eq. (1)]:

$$\left|\begin{array}{c} I_{1a} \\ I_{1b} \\ I_{2a} \\ I_{2b} \\ I_{3a} \\ I_{3b} \\ \cdot \\ \cdot \\ \cdot \\ I_{na} \\ I_{nb} \end{array}\right| = p \left|\begin{array}{ccccccccccc} Y_{11} & -Y_{11}t & Y_{12} & -Y_{12}t & 0 & \cdot & \cdot & & & & \\ -Y_{11}t & Y_{11} & -Y_{12}t & Y_{12} & 0 & \cdot & \cdot & & & & \\ Y_{12} & -Y_{12}t & Y_{22} & -Y_{22}t & Y_{23} & -Y_{23}t & 0 & \cdot & \cdot & & \\ -Y_{12}t & Y_{12} & -Y_{22}t & Y_{22} & -Y_{23}t & Y_{23} & 0 & \cdot & \cdot & & \\ 0 & 0 & Y_{23} & -Y_{23}t & Y_{33} & -Y_{33}t & Y_{34} & -Y_{34}t & 0 & \cdot & \cdot \\ \cdot & \cdot & -Y_{23}t & Y_{23} & -Y_{32}t & Y_{33} & -Y_{34}t & Y_{34} & 0 & \cdot & \cdot \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Y_{nn} & \cdot & -Y_{nn}t \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -Y_{nn}t & \cdot & Y_{nn} \end{array}\right| \cdot \left|\begin{array}{c} V_{1a} \\ V_{1b} \\ V_{2a} \\ V_{2b} \\ V_{3a} \\ V_{3b} \\ \cdot \\ \cdot \\ \cdot \\ V_{na} \\ V_{nb} \end{array}\right| \quad (1)$$

where $p = -j \cot(\theta)$ and $t = \sec(\theta)$.

From the definition of the admittance matrix

$$y_{i,j}^{u,v} = \left. \frac{I_{j,v}}{V_{i,u}} \right|_{\text{all voltages are zero except } V_{i,u}} \quad (2)$$

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where $I_{j,v}$ is the current flowing into the jv node and $V_{i,u}$ is the voltage of node iu with respect to the ground plane. i and j take on values from 1 through n, u and v are either a or b. Reciprocity requires that $y_{i,j}^{u,v} = y_{j,i}^{v,u}$ and hence we can find all of the element values of the admittance matrix as given in (2) by connecting a voltage source to only one side of each line, i.e., to nodes ia as shown in Fig. 2. Symmetry requires that $y_{i,j}^{u,v} = y_{i,j}^{v,u}$ and $y_{i,j}^{u,u} = y_{i,j}^{v,v}$.

We will solve the problem of Fig. 2 by superposition of the circuit of Fig. 3(a) and the circuit of Fig. 3(b). We will refer to the excitation of Fig. 3(a) as mode 1 excitation and that of Fig. 3(b) as mode 2 excitation. Under mode 1 excitation the incident voltage wave on all of the lines will be equal. Since all of the lines are terminated in a short circuit the reflected waves will also be equal. Thus the voltage on all of the lines will be equal for their entire length. It is well known that for a wave propagating in the TEM mode the fields in the transverse plane satisfy Laplace's equation, and hence the characteristic impedances can be determined from a static field configuration or from the static capacitances. In Fig. 4(a) the static capacitances of the coupled lines are shown. For the mode 1 excitation the voltage on all of the lines are equal and hence there can be no current through the mutual capacitances. We can treat the lines as uncoupled lines with capacitance to ground per unit length equal to C_i. Under mode 2 excitation the incident voltage on line i will be the negative of the voltage on all of the other lines. Since all of the lines have short-circuited terminations the reflected voltage wave

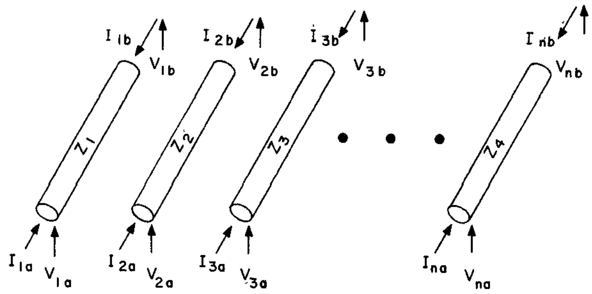

Fig. 1.

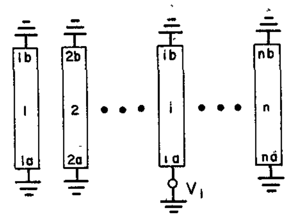

Fig. 2.

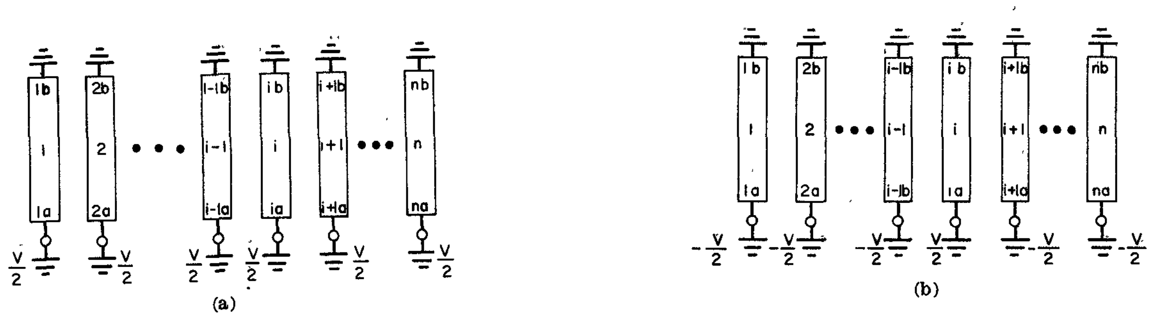

Fig. 3.

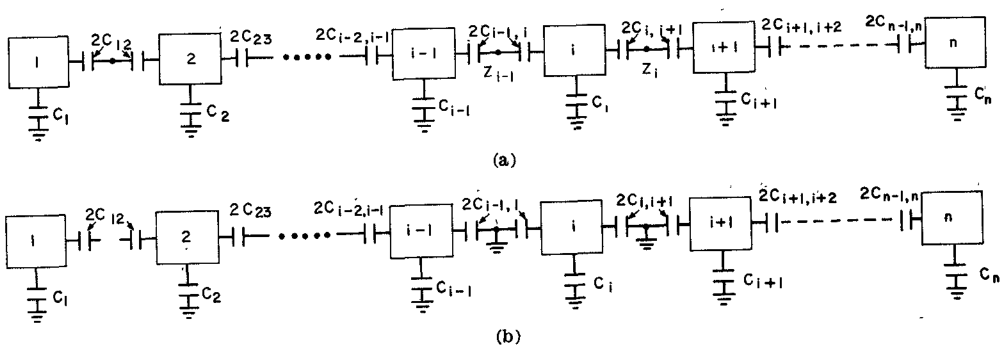

Fig. 4.

on line i will be the negative of the reflected waves on all of the other lines. Thus the voltage on line i will be the negative of the voltage on all of the other lines for its entire length. With reference to Fig. 4(a), if line i is at a potential V and line $i-1$ and lines $i+1$ are at potential $-V$ then points z_{i-1} and z_i must be at zero potential and hence can be grounded. The static capacitances for mode 2 excitation are therefore those shown in Fig. 4(b).

Using the notation that Y_{01}^j is the mode 1 characteristic admittance of the jth line, and Y_{02}^j is the mode 2 characteristic admittance of the jth line, we can write

$$Y_{01}^j = \nu C_j, \qquad j = 1,2,\cdots n$$

$$Y_{02}^{i-1} = \nu(C_{i-1} + 2C_{i-1,i}), \qquad i = 1,2,\cdots n$$

$$Y_{02}^i = \nu(C_i + 2C_{i-1,i} + 2C_{i,i+1}), \qquad i = 1,2,\cdots n$$

$$Y_{02}^{i+1} = \nu(C_{i+1} + 2C_{i,i+1}), \qquad i = 1,2,\cdots n$$

$$Y_{02}^j = \nu C_j, \qquad j = 1,2,\cdots i-2, i+2,\cdots n \tag{3}$$

where

$$C_{01} = C_{n,n+1} = 0 \qquad C_0 = C_{n+1} = 0.$$

We shall now proceed to find the currents in all of the nodes for mode 1 and mode 2 excitations. Adding the respective currents will then yield the currents in all of the nodes of the circuit of Fig. 2. Equation (2) then yields all of the matrix elements. We treat all of the lines as uncoupled lines with characteristic admittances given by (3). The currents are given by

$$I_{i,a}^1 = -jV_1 \cot(\theta) Y_{01}^i/2 \qquad I_{i,a}^2 -jV_1(\cot(\theta) Y_{02}^i/2$$

$$I_{i,b}^1 = jV_1 Y_{01}^i/(2\sin(\theta)) \qquad I_{i,b}^2 = jV_1 Y_{02}^i/(2\sin(\theta))$$

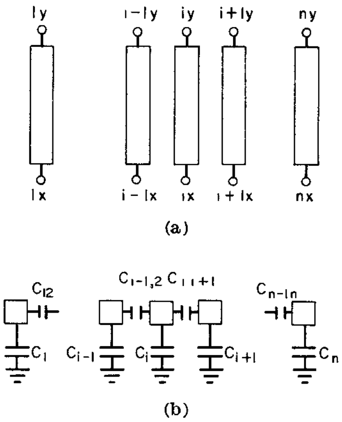

Fig. 1. (a) n coupled transmission lines. (b) Capacitances per unit length of the lines of Fig. 1(a).

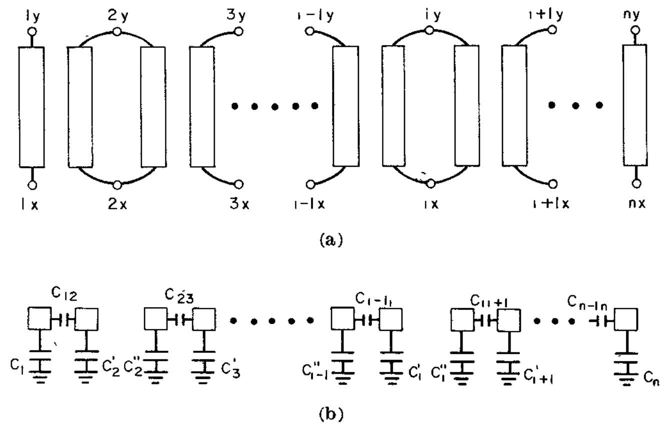

Fig. 2. (a) Network consisting of $n - 1$ pairs of coupled lines. (b) Capacitances per unit length of the lines of Fig. 2(a).

$$
\begin{aligned}
I_{i+1,a}^1 &= -jV_1 \cot(\theta) Y_{01}^{i+1}/2 & I_{i+1,a}^2 &= jV_1 \cot(\theta) Y_{02}^{i+1}/2 \\
I_{i+1,b}^1 &= jV_1 Y_{01}^{i+1}/(2\sin(\theta)) & I_{i+1,b}^2 &= -jV_1 Y_{02}^{i+1}/(2\sin(\theta)) \\
I_{i-1,a}^1 &= -jV_1 \cot(\theta) Y_{01}^{i-1}/2 & I_{i-1,a}^2 &= jV_1 \cot(\theta) Y_{02}^{i-1}/2 \\
I_{i-1,b}^1 &= jV_1 Y_{01}^{i-1}/(2\sin(\theta)) & I_{i-1,b}^2 &= -jV_1 Y_{02}^{i-1}/(2\sin(\theta))
\end{aligned}
\tag{4}
$$

where $I_{j,v}^1$ is the current in the jv node for mode 1 excitation and similarly $I_{j,v}^2$ is the current in the jv node for mode 2 excitation. Equation (2) then yields, when $p = -j\cot(\theta)$ and $t = \sec(\theta)$,

$$
\begin{aligned}
y_{i,i}^{a,a} &= -j\cot(\theta)(Y_{01}^i + Y_{02}^i)/2 = \nu(C_i + C_{i-1,i} + C_{i,i+1})p \\
y_{ii}^{a,b} &= j(Y_{01}^i + Y_{02}^i)/(2\sin(\theta)) = -\nu(C_i + C_{i-1,i} + C_{i,i+1})pt \\
y_{i,i+1}^{a,a} &= j\cot(\theta)(Y_{02}^{i+1} - Y_{01}^{i+1})/2 = -\nu(C_{i,i+1})p \\
y_{i,i+1}^{a,b} &= -j(Y_{02}^{i+1} - Y_{01}^{i+1})/(2\sin(\theta)) = \nu(C_{i,i+1})pt \\
y_{i,i-1}^{a,a} &= j\cot(\theta)(Y_{02}^{i-1} - Y_{01}^{i-1})/2 = -\nu(C_{i-1,i})p \\
y_{i,i-1}^{a,b} &= -j(Y_{02}^{i-1} - Y_{01}^{i-1})/(2\sin(\theta)) = \nu(C_{i-1,i})pt.
\end{aligned}
\tag{5}
$$

We see from (3) that if $|i - j|$ is greater than 1 that $Y_{01}^j = Y_{02}^j$ and hence superposition of mode 1 and mode 2 excitation yields zero current, and $y_{i,j}^{u,v} = 0$. We find using (5), reciprocity and the symmetry of the network, that the admittance matrix is that of (1) where

$$
\begin{aligned}
Y_{11} &= \nu(C_1 + C_{12}) \\
Y_{ii} &= \nu(C_{i-1,i} + C_i + C_{i,i+1}) \\
Y_{nn} &= \nu(C_n + C_{n-1,n}) \\
Y_{i,i+1} &= -\nu C_{i,i+1}.
\end{aligned}
\tag{6}
$$

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A Useful Identity for the Analysis of a Class of Coupled Transmission-Line Structures

A. I. GRAYZEL

Abstract—**In this letter an identity is proven which allows easy analysis of many coupled transmission-line structures.**

In this letter we will prove the following theorem: if only nearest neighbor couplings are considered n commensurate-coupled transmission lines can be reduced to a network consisting of $n - 1$ pairs of coupled lines.

To prove this theorem we will prove that the network of Fig. 1 and the network of Fig. 2 are equivalent. In Fig. 1(a) is shown a

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